

# Moments from their very truncations

Franciszek Hugon Szafraniec

*Andrzej Lasota (1932 – 2006) in memoriam*

**ABSTRACT.** It is known that positive definiteness is not enough for the multidimensional moment problem to be solved. We would like throw in to the garden of existing in this matter so far results one more, a result which takes into considerations the utmost possible truncations.

As we have already pointed out positive definiteness is not sufficient for a multisequence to be a moment one, neither in the case of real moment problem in more than one variable, nor for a complex one or any complex dimension; for the previous one we recommend the cult paper of Fuglede [5], for the second mentioned [8] can be regarded as a source of information. Replacing it by solvability of a kind of truncations we gain necessary and sufficient conditions for the moment problem to be settled. It is worthy to say that truncations in the multivariable development have been considered from diverse points of view; let us have [3], [4], [7], [16] or [17] as a choice of references.

**1.** Let  $\mathcal{M}(X)$  stand for the space of all regular complex Borel measures on a locally compact space  $X$  and let  $\mathcal{M}_a(X)$  be the collection of all positive measures in  $\mathcal{M}(X)$  such that  $\mu(X) = a$ . Consider  $\mathcal{M}(X)$  with the  $\sigma(\mathcal{M}(X), \mathcal{C}_b(X))$  topology<sup>1</sup>, where  $\mathcal{C}_b(X)$  is the Banach space of continuous and bounded functions on  $X$  with the ‘sup’ norm; the topology is determined by the duality

$$(\mu, f) \mapsto \int_X f \, d\mu, \quad \mu \in \mathcal{M}(X), \quad f \in \mathcal{C}_b(X).$$

One of the pleasant features of the  $\sigma(\mathcal{M}(X), \mathcal{C}_b(X))$  topology is that  $\mathcal{M}_a(X)$  is stable under the closure, another is that it coincides on  $\mathcal{M}_a(X)$  with the  $*$ -weak topology.

---

1991 *Mathematics Subject Classification.* Primary 44A60; Secondary 43A35, 43A05, 47B32, 47B15, 47B20.

*Key words and phrases.* multidimensional moment problem, complex moment problem, truncated moment problem, symmetric operator, selfadjoint operator, spectral measure, semispectral measure, elementary spectral measure, sesquilinear selection, Jordan–von Neumann theorem, Markoff–Kakutani theorem.

<sup>1</sup> This topology is called ‘weak’ one in Probability and Stochastic Processes though it may sound oddly for people in Functional Analysis.

Let  $\Xi$  be a linear space with a seminorm  $p$ . Set

$$\mathcal{M}_\Xi \stackrel{\text{df}}{=} \prod_{\xi \in \Xi} \mathcal{M}_\xi,$$

where  $\mathcal{M}_\xi = \mathcal{M}(X)$  for every  $\xi \in \Xi$ . Endow  $\mathcal{M}_\Xi$  with the Tychonoff topology based on that of  $\mathcal{M}(X)$ . Having  $\{\mu_\xi\}_\xi \in \mathcal{M}_\Xi$  define

$$\mu_{\xi,\eta} \stackrel{\text{df}}{=} \frac{1}{4}(\mu_{\xi+\eta} - \mu_{\xi-\eta} + i\mu_{\xi+i\eta} - i\mu_{\xi-i\eta}), \quad \xi, \eta \in \Xi. \quad (1)$$

The following selection result is in <sup>2</sup> [11].

**THEOREM 1.** *Let  $\mathcal{M} \subset \mathcal{M}_\Xi$  be a nonempty set and let  $p$  be a seminorm satisfying the parallelogram law*

$$p(\xi + \eta)^2 + p(\xi - \eta)^2 = 2(p(\xi)^2 + p(\eta)^2), \quad \xi, \eta \in \Xi.$$

*Suppose  $\{\mu_f\}_f \in \mathcal{M}$  implies*

- (i)  $\{\mu_{i\xi}\}_\xi \in \Xi$  as well as  $\{t^{-2}\mu_{t\xi}\}_\xi \in \mathcal{M}$  for  $t \in \mathbb{R} \setminus \{0\}$  and  $\{\frac{1}{2}\mu_{\xi+\eta} + \frac{1}{2}\mu_{\xi-\eta} - \mu_\eta\}_\xi \in \mathcal{M}$  for  $\eta \in \Xi$ ;
- (ii)  $\mu_0 = 0$ ,  $\mu_{\xi+\eta} + \mu_{\xi-\eta} - 2\mu_\eta \geq 0$ ;
- (iii)  $\mu_\xi(X) = p(\xi)^2$ ,  $\xi \in \Xi$ .

*Then there is  $\{\mu_\xi\}_\xi \in \text{cloconv}(\mathcal{M})$  such that*

$$\mu_\xi \geq 0, \quad \mu_{\xi+\eta} + \mu_{\xi-\eta} = 2(\mu_\xi + \mu_\eta), \quad \mu_{z\xi} = |z|^2 \mu_\xi, \quad \xi, \eta \in \Xi, \quad z \in \mathbb{C}. \quad (2)$$

*Consequently, for every Borel subset  $\sigma$  of  $X$  the mapping, cf. (1),*

$$(\xi, \eta) \rightarrow \mu_{\xi,\eta}(\sigma) \quad (3)$$

*is a positive Hermitian bilinear (=positive sesquilinear) form on  $\Xi$  and*

$$\mu_{\xi,\xi}(X) = p(\xi)^2, \quad \xi \in \Xi. \quad (4)$$

The proof relies on Markoff–Kakutani fixed point theorem.

*Remark 2.* If  $\{\mu_\xi\}_\xi$  is as in the conclusion of Theorem 1 then

$$|\mu_{\xi,\eta}(\sigma)| \leq p(\xi)p(\eta), \quad \xi, \eta \in \Xi \quad \sigma \text{ a Borel subset of } X.$$

Indeed, by the Schwarz inequality applied to the mapping of (3) and then by (4) we have (apparently  $\mu_\xi = \mu_{\xi,\xi}$  due to (1) and (2))

$$|\mu_{\xi,\eta}(\sigma)|^2 \leq \mu_\xi(\sigma)\mu_\eta(\sigma) \leq \mu_\xi(X)\mu_\eta(X) \leq p(\xi)^2 p(\eta)^2.$$

---

<sup>2</sup> Actually it is stated and proved there for the  $*$ -weak topology in  $\mathcal{M}(X)$ , however the proof brings over *verbatim* to the  $\sigma(\mathcal{M}(X), \mathcal{C}_b(X))$  topology case. As an immediate consequence we can replace inequality in condition (iii) of [11] by equality and this results in equality in (b) here (notice that due to (ii) the measures involved are positive). Also we replace norm by seminorm which is still acceptable due to our general version of Jordan–von Neumann Theorem therein. The method is flexible enough to tolerate all these changes.

**2.** For  $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ ,  $N = \{0, 1, 2, \dots\}$  here, and for  $x = (x_1, \dots, x_d) \in \mathbb{R}$  or  $\mathbb{C}$  we hold up the notation:  $|n| \stackrel{\text{df}}{=} n_1 + \dots + n_d$  and  $x^n \stackrel{\text{df}}{=} x_1^{n_1} \dots x_d^{n_d}$ . Moreover, a bit perversely,  $\infty \stackrel{\text{df}}{=} (\infty, \dots, \infty)$ . Notation for the basic zero-one  $d$ -tuples is shortened to

$$e_i \stackrel{\text{df}}{=} (\delta_{m,i})_{m=1}^d, \quad i = 1, \dots, d. \quad (5)$$

A  $d$ -sequence  $(a_n)_{n=0}^\infty$ ,  $a_n = a_{n_1, \dots, n_d}$ , is said to be ( $d$ -dimensional *real*) *moment* one if there is a positive measure  $\mu$  on  $\mathbb{R}^d$  such that

$$a_n = \int_{\mathbb{R}^d} x^n \mu(dx), \quad n \in \mathbb{N}^d.$$

Likewise,  $(c_{m,n})_{m,n=0}^\infty$  is said to be a ( $d$ -dimensional *complex*) *moment*  $2d$ -sequence if there is a positive measure  $\nu$  on  $\mathbb{C}^d$  such that

$$c_{m,n} = \int_{\mathbb{C}^d} z^m \bar{z}^n \nu(dz), \quad m, n \in \mathbb{N}^d.$$

Apparently there are two definitions of *positive definiteness*: for  $(a_n)_{n=0}^\infty$  as well as for  $(c_{m,n})_{m,n=0}^\infty$ . These multisequences have to satisfy the following conditions

$$\sum_{m,n} a_{m+n} \xi_m \bar{\xi}_n \geq 0 \text{ for any finite sequence of } (\xi_n)_n \subset \mathbb{C} \quad (\text{PD}_{\mathbb{R}})$$

or

$$\sum_{m,n,k,l} c_{m+l,n+k} \xi_{m,n} \bar{\xi}_{k,l} \geq 0 \text{ for any finite sequence of } (\xi_{m,n})_{m,n} \subset \mathbb{C}. \quad (\text{PD}_{\mathbb{C}})$$

It is commonly known that positive definiteness is sufficient for a multisequence to be a moment one but it is not necessary except the 1-dimensional real case. Our goal here is to present necessary and sufficient conditions for the moment problem to be solved.

Let  $\Xi$  denotes the linear space of all  $d$ -sequences  $(\xi_n)_{|n| \geq 0}$  of complex numbers which are zero but a finite number.

For  $\xi = (\xi_n)_n \in \Xi$  define new multisequences  $(a_n^\xi)_{n=0}^\infty$  and  $(c_{m,n}^\xi)_{m,n=0}^\infty$  as follows

$$a_n^\xi \stackrel{\text{df}}{=} \sum_{k,l} a_{n+k+l} \xi_k \bar{\xi}_l, \quad c_{m,n}^\xi \stackrel{\text{df}}{=} \sum_{k,l} c_{m+k,n+l} \xi_k \bar{\xi}_l, \quad m, n \in \mathbb{N}. \quad (6)$$

Let us set

$$\mathbf{N}_1 \stackrel{\text{df}}{=} \{k: |k| \leq 1\}.$$

**THEOREM 3.** A  $d$ -sequence  $(a_n)_{n=0}^\infty$  is a moment one if and only if there is a family  $\{\mu_\xi\}_{\xi \in \Xi}$  of positive measures on  $\mathbb{R}^d$  such that

$$a_n^\xi = \int_{\mathbb{R}^d} x^n \mu_\xi(dx), \quad n \in \mathbf{N}_1 \cup 2\mathbf{N}_1 \quad (7)$$

and

$$\mu_0 = 0, \quad \mu_{\xi+\eta} + \mu_{\xi-\eta} - 2\mu_\eta \geq 0, \quad \xi, \eta \in \Xi. \quad (8)$$

The twin result, concerning the complex moment problem is as follows

**THEOREM 4.** A  $2d$ -sequence  $(c_{m,n})_{m,n=0}^\infty$  is a moment one if and only if there is a family  $\{\mu_\xi\}_{\xi \in \Xi}$  of positive measures on  $\mathbb{C}^d$  such that

$$c_{m,n}^\xi = \int_{\mathbb{C}^d} z^m \bar{z}^n \mu_\xi(dz), \quad m, n \in \mathbf{N}_1$$

and

$$\mu_0 = 0, \quad \mu_{\xi+\eta} + \mu_{\xi-\eta} - 2\mu_\eta \geq 0, \quad \xi, \eta \in \Xi.$$

There is a rather formal link between  $2d$ -dimensional real moment problem and  $d$ -dimensional complex one (see, Proposition 57 in [8]) however on the operator level, which is pretty often used in proofs, it becomes fragile. Fortunately, the proof of Theorem 4 goes the same way as that of Theorem 3 otherwise a reader may elaborate the aforesaid link for him/herself.

**3. PROOF OF THEOREM 3.** Putting  $n = 0$  in (7) we get  $(\text{PD}_{\mathbb{R}})$ . This allows us to construct a reproducing kernel Hilbert space<sup>3</sup>  $\mathcal{H}$ , say, composed of sequences of complex numbers. More precisely, after defining  $a_{(n)} \stackrel{\text{df}}{=} (a_{n+k})_k$  the scalar product of  $\mathcal{H}$  is given as

$$\langle a_{(m)}, a_{(n)} \rangle \stackrel{\text{df}}{=} a_{m+n}, \quad m, n \in \mathbb{N}^d.$$

The space  $\mathcal{D} \stackrel{\text{df}}{=} \text{lin}\{a_{(n)} : n \in \mathbb{N}^d\}$  is dense in  $\mathcal{H}$  and

$$a_{m+n}^\xi = \langle \sum_k \xi_k a_{(m+k)}, \sum_l \xi_l a_{(n+l)} \rangle, \quad m, n \in \mathbb{N}^d. \quad (9)$$

Define on  $\Xi$  a seminorm  $p$  by

$$p(\xi) = \left\| \sum_k \xi_k a_{(k)} \right\|, \quad \xi = (\xi_k)_k \in \Xi$$

and by  $\langle \cdot, - \rangle_p$  the related semi-inner product.

*Remark 5.* It is clear that  $p(\xi' - \xi'') = 0$  if and only if  $\xi' - \xi'' \in \Delta$  where

$$\Delta \stackrel{\text{df}}{=} \{(\xi_k)_k \in \Xi : \sum_k \xi_k a_{m+k} = 0 \text{ for all } m \in \mathbb{N}^d\} = \{(\xi_k)_k \in \Xi : \sum_k \xi_k a_{(k)} = 0\}.$$

The set  $\Xi$  is apparently a linear subspace of  $\Xi$  and so is  $\hat{\Xi} \stackrel{\text{df}}{=} \Xi / \Delta$ . Moreover,  $\hat{\Xi}$  is a unitary space, the mapping

$$h : \hat{\Xi} \ni \hat{\xi} \mapsto \sum_k \xi_k a_{(k)} \in \mathcal{D}$$

is well defined and becomes a unitary operator between  $\hat{\Xi}$  and  $\mathcal{D}$

If  $\mathcal{M}$  is defined by means of all  $\mu_\xi$ ,  $\xi \in \Xi$ , determined by (7) and (8), then there is a matter of direct verification to check that all the assumptions of Theorem 3 are fulfilled; in particular  $\mathcal{M}$  is non empty (notice that (8) with  $\xi = 0$  forces the measures  $\mu_\xi$  to be positive). Now it is a right time to make use of Theorem 1. However, before doing this notice that  $\mathcal{M}$  is already a convex set. So we have got a family  $\{\mu_\xi\}_\xi$  of positive measure such that

$$\{\mu_\xi\}_\xi \in \text{clo}(\mathcal{M}). \quad (10)$$

and (3) as well as (4) hold true. In particular, sesquilinearity in condition (3) of Theorem 1 supported by Remark 2 allows us to define the family  $\{\mu_f\}_{f \in \mathcal{D}}$  by  $\mu_f \stackrel{\text{df}}{=} \mu_{h(\hat{\xi})}$  which is a well defined measure as long as  $\xi \in h^{-1}(f) = \hat{\xi} \in \hat{\Xi}$ , cf.

---

<sup>3</sup> Though elements of the RKHS approach can be traced on many occasions we would like to advertise here [14], at least for those who can read it.

Remark 5<sup>4</sup>. This brings us back to the Hilbert space  $\mathcal{H}$  with (3) to be satisfied after replacing  $\xi, \eta$  by  $f, g$  and the semi-norm in (4) to be the norm of  $\mathcal{H}$ . Now by standard means we extend  $\mu_{f,g}$  to the whole of  $\mathcal{H}$  and find a semispectral measure  $F$  in  $\mathcal{H}$  such that

$$\mu_{f,g} = \langle F(\cdot)f, g \rangle, \quad f, g \in \mathcal{H}.$$

With the shorthand notation (5) in mind define the operators  $A_i$  with domain  $\mathcal{D}(A^i) \stackrel{\text{df}}{=} \mathcal{D}$  by  $A_i a_{(m)} \stackrel{\text{df}}{=} a_{(m+e_i)}$ . The operators  $A_i$  are symmetric,  $\mathcal{D}$  is invariant for each of them and they commute pointwise on  $\mathcal{D}$ . With  $A^n \stackrel{\text{df}}{=} A_1^{n_1} \cdots A_d^{n_d}$  we have by (9)

$$a_n^\xi = \langle A^n \sum_k \xi_k a_{(k)}, \sum_l \xi_l a_{(l)} \rangle, \quad n \in \mathbb{N}^d, \quad \xi \in \Xi. \quad (11)$$

Due to (10), after all those identifications, we can say for any  $f \in \mathcal{D}$  there is a net  $\{\mu_f^\alpha\}_\alpha \subset \mathcal{M}$  such that

$$\int_{\mathbb{R}^d} \varphi d\mu_f^\alpha \rightarrow \int_{\mathbb{R}^d} \varphi d\langle F(\cdot)f, f \rangle, \quad \varphi \in \mathcal{C}_b(\mathbb{R}^d). \quad (12)$$

Let  $E$  be a spectral measure which the Naimark dilation of  $F$  living presumably in a larger space  $\mathcal{K}$  and let  $B_i$  be defined as

$$\langle B_i x, y \rangle_{\mathcal{K}} \stackrel{\text{df}}{=} \int_{\mathbb{R}^d} t^i \langle E(dt)x, y \rangle_{\mathcal{K}}, \quad x \in \mathcal{D}(B), \quad y \in \mathcal{H}, \quad |i| = 1$$

each with its maximal domain. We want to know  $B_i$ 's are selfadjoint extensions of  $A_i$ 's. For this we use Lemma 6 twice.

First we show that  $\mathcal{D} = \mathcal{D}(A_i) \subset \mathcal{D}(B_i)$ ,  $i = 1, \dots, d$ . For this the working part of condition 1<sup>o</sup> of Lemma 6 applied to  $\Phi(t) \equiv |t_i|^2$  reads as

$$\int_{\mathbb{R}^d} |t_i|^2 \mu_f^\alpha(dt) = a_{2e_i}^f \implies \int_{\mathbb{R}^d} |t_i|^2 \langle E(dt)f, f \rangle_{\mathcal{K}} \leq a_{2e_i}^f$$

which means that  $f$  is in the domain of  $B_i$ . Now according to the definition of  $\mathcal{M}$  we have, by 2<sup>o</sup> of Lemma 6,

$$\langle A_i f, f \rangle - \langle B_i f, f \rangle = \int_{\mathbb{R}^d} x_i d\mu_f^\alpha - \int_{\mathbb{R}^d} x_i \langle E(dx)f, f \rangle = 0$$

as well as

$$\langle A_i f, A_i f \rangle - \langle B_i f, B_i f \rangle = \int_{\mathbb{R}^d} |t_i|^2 \mu_f^\alpha(dt) - \int_{\mathbb{R}^d} |t_i|^2 \langle E(dt)f, f \rangle = 0$$

All this gives us  $A_i \subset B_i$  for  $i = 1, \dots, d$ , cf. [15, §5].

Now, because  $\mathcal{D}$  is invariant for every  $A^n$  it is so also for  $B^n$ . Thus, due to (11), using multiplicativity properties of spectral integrals (see, Theorem 4, p. 135

---

<sup>4</sup> The reason we have had to make the quotient operation at this stage is not because the proof would not work. It is grounded upon incomparably deeper circumstances. For the 1-dimensional real case people pretty often neglect this assuming additionally that  $\Delta = \{0\}$ . While this does not cause too much pain (except aesthetical discomfort), in the others it may generate great loss: the would-be moment measures supported on real algebraic sets, regardless the moment problem itself is real or complex, may be out of game – those important measures are just encoded in  $\Delta$ , for much more on this look at [2] and [13]. The RKHS approach shows sign of its might!

in  $[1]$ ),

$$a_n = \langle A^n a_{(0)}, a_{(0)} \rangle = \langle B^n a_{(0)}, a_{(0)} \rangle = \int_{\mathbb{R}^d} x \langle E(dx) a_{(0)}, a_{(0)} \rangle = \int_{\mathbb{R}^d} x^n \mu(dx),$$

$n \in \mathbb{N}^d$

with  $\mu = \langle F(\cdot) a_{(0)}, a_{(0)} \rangle$ . Thus  $(a_n)_n$  is a  $d$ -dimensional moment  $d$ -sequence according to our wish.

The ‘only if’ part is a matter of straightforward verification.  $\square$

**4.** Let us prove the main ingredient of the above proof because it may be interesting and useful for itself. For a topological space  $X$  denote by  $\mathcal{C}_c(X)$  the space of all continuous complex functions with compact support.

LEMMA 6. *Let  $X$  be a locally compact Polish space. If  $\phi$  is a continuous complex function on  $X$ ,  $\{\mu_\alpha\}_\alpha$  is a net in  $\mathcal{M}_1(X)$  and  $\mu$  is a positive measure. Consider the limit*

$$\lim_\alpha \int_X \phi d\mu_\alpha = \int_X \phi d\mu. \quad (13)$$

1° *If (13) holds for every  $\phi \in \mathcal{C}_c(X)$  and  $\int_X \phi d\mu_\alpha \leq c$  uniformly in  $\alpha$  then  $\int_X \phi d\mu \leq c$  provided  $\phi \geq 0$ ;*

2° *if (13) holds for every  $\phi \in \mathcal{C}_b(X)$  and  $\int_X \phi d\mu_\alpha \rightarrow a$  then  $\int_X \phi d\mu = a$  provided  $\phi$  is such that  $\int_X |\phi|^2 d\mu_\alpha \leq c$  uniformly in  $\alpha$ .*

PROOF. Consider a sequence  $(\varphi_k)_k \in \mathcal{C}_c(X)$  such that  $0 \leq \varphi_k \nearrow 1$  pointwise. Then the sequence  $\{\text{supp } \varphi_k\}_k$  of compact sets nests  $X$ .

Taking the limit passage of the left hand side of

$$\int_X \varphi_k \phi d\mu_\alpha \leq \int_X \phi d\mu_\alpha \leq c$$

first in  $\alpha$  then in  $k$  we come up to 1°.

For 2° take  $\varphi_k(z) = 1$  and write

$$\begin{aligned} |a - \int_X \phi d\mu| &\leq |a - \int_X \phi d\mu_\alpha| + \left| \int_X (1 - \varphi_k) \phi d\mu_\alpha \right| \\ &\quad + \left| \int_X \varphi_k \phi d(\mu_\alpha - \mu) \right| + \left| \int_X (\varphi_k - 1) \phi d\mu \right|. \end{aligned}$$

The second term plays the most sensitive role so let us treat it as follows.

By the theorem of Prokhorov ([6], theorem 6.7, p.47 or [10], p. 121) for a given  $\varepsilon$  there is a compact subset  $K$  of  $X$  such that  $\mu_\alpha(X \setminus K) < \varepsilon$  for any  $\alpha$ . Now, pick up  $k_0$  so that  $\varphi_k = 1$  on  $K$  for  $k > k_0$  and write

$$\begin{aligned} \left| \int_X (1 - \varphi_k) \phi d\mu_\alpha \right| &\leq \left| \int_K (1 - \varphi_k) \phi d\mu_\alpha \right| + \left| \int_{X \setminus K} (1 - \varphi_k) \phi d\mu_\alpha \right| \\ &\leq \left| \int_K (1 - \varphi_k) \phi d\mu_\alpha \right| \\ &\quad + \sqrt{\int_{X \setminus K} |1 - \varphi_k|^2 d\mu_\alpha \int_{X \setminus K} |\phi|^2 d\mu_\alpha} \\ &= \sqrt{\varepsilon c}. \end{aligned}$$

Notice the evaluation holds for all  $\alpha$ 's uniformly in  $k > k_0$ . Because of this we can start with evaluating the forth term going beyond  $k_0$ , if necessary, and being backed by the Schwarz inequality

$$\left| \int_X (\varphi_k - 1) \phi \, d\mu \right|^2 \leq \int_X |\varphi_k - 1| \, d\mu \int_X |\phi|^2 \, d\mu$$

and 1°, then fixing  $k$  in the third make this, fix  $\alpha$  in the first and finally take the advantage of the evaluation for the second.  $\square$

**5.** We admit that the machinery we have used is pretty heavy. This is so because we have patterned our proof on the content of [11] and that concerns operators. We may have a hope it can be done in more direct way and this may be a kind of challenge.

At first glance it looks like our result is of different nature than of [9]. Instead of solving all the truncated moment problems coming from a given multisequence, as required in [9], we confine ourselves to the (family of) the very initial truncations. As both approaches provide us with necessary and sufficient conditions they stimulate a question of comparing their usefulness. If one agrees a truncated moment problem should be solved in finitely supported measures to make things easier, our truncations lead to algebraic conditions of order at most 2; however there is a family of them to be solved, all of them subject to the constraints of the type (8). We count on the invitation to be accepted.

*Aknowlegments.* The main result of this paper was mentioned on several occasions. It was put forward for the first time already during the combined 1996 Iowa events: *Workshop on Recent Developments in Moments and Operators* and *AMS Special Session on Moments and Operators*; the last presentation was at the Edwardsville 2006 conference, since the latter author's glimmer of hope to give it a matured written form has materialized in this account.

The author would like to acknowledge his appreciation of Jan Stochel's remarks on the previous version of this paper.

## References

1. M.S. Birman, M.Z. Solomjak, *Spectral theory of self-adjoint operators in Hilbert space*, D. Reidel Publishing Company, Dordrecht/Boston/Lancaster/Tokyo, **1987**.
2. D. Cichoń, J. Stochel and F.H. Szafraniec, Three term recurrence relation modulo an ideal and orthogonality of polynomials of several variables, *J. Approx. Theory*, **134** (2005), 11–64.
3. R.E. Curto, L. Fialkow, Solution of the truncated complex moment problem for flat data, *Memoirs Amer. Math. Soc.*, **568** (1996).
4. ———, A duality proof of Tchakaloff's theorem, *J. Math. Anal. Appl.*, **269** (2002), 519–532.
5. B. Fuglede, The multidimensional moment problem, *Expo. Math.*, **1** (1983), 47–65.
6. K.R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York–London, **1962**.
7. M. Putinar, A dilation theory approach to cubature formulas, *Expo. Math.*, **15** (1997), 183–192.
8. J. Stochel, F. H. Szafraniec, The complex moment problem and subnormality: a polar decomposition approach, *J. Funct. Anal.* **159** (1998), 432–491.
9. J. Stochel, Solving the truncated moment problem solves the full moment problem, *Glasgow J. Math.*, **43** (2001), 335–341.
10. D. Strook, *Probability Theory, an analytic view*, Cambridge Univ. Press, Cambridge, UK, **1994**.
11. F.H. Szafraniec, Sesquilinear selection of elementary spectral measures and subnormality, in *Elementary Operators and Applications*, Proceedings, Blaubeuren bei Ulm (Deutschland), June 9–12, 1991, ed. M.Mathieu, pp. 243–248, World Scientific, Singapore, **1992**.

12. ———, Subnormality and cyclicity, *Banach Center Publications*, **67** (2005), 349–356.
13. ———, Favard's theorem modulo an ideal, *Oper. Theory Adv. Appl.*, **157** (2005), 301–310.
14. ———, *Reproducing kernel Hilbert spaces* [in Polish], Wydawnictwo Uniwersytetu Jagiellońskiego, Kraków, **2004**.
15. B. Sz.-Nagy, *Extensions of linear transformations in Hilbert space which extend beyond this space*, Appendix to F.Riesz, B.Sz.-Nagy, *Functional Analysis*, Ungar, New York, **1960**
16. Y. Xu, Cubature formulae and polynomial ideals, *Adv. Appl. Math.*, **23** (1999), 211–233.
17. ———, Constructing cubature formulae by the method of reproducing kernel, *Numer. Math.*, **85** (2000), 155–173.

INSTYTUT MATEMATYKI, UNIWERSYTET JAGIELLOŃSKI, UL. REYMONTA 4, PL-30059 KRAKÓW  
E-mail address: [Franciszek.Szafraniec@im.uj.edu.pl](mailto:Franciszek.Szafraniec@im.uj.edu.pl)